1-dimensional Representations and abelian groups

Schur's lemma implies:

$\varphi : G \longrightarrow GL(V)$ irreducible.

$T : V \to V$ such that $\varphi_g T = T \varphi_g$ $\forall g \in G$

$\Longrightarrow$ $T = \lambda I$ for some $\lambda \in \mathbb{C}$

Prop: G - finite abelian group

Then every irreducible rep of G is 1 dimensional

Proof: $\varphi : G \to GL(V)$ irreducible.

$g, h \in G \Rightarrow \varphi_g, \varphi_h : V \to V$, $\varphi_g \varphi_h = \varphi_{gh}$

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Let $T = \varphi_h \in \mathrm{Hom}_G(\varphi, \varphi)$

Schur $\Rightarrow$ $T = \lambda I$

i.e, $\varphi_h = \lambda_h I$ for some $\lambda_h \in \mathbb{C}$

$\forall h \in G$

pick $v \in V$, $v \neq 0$, $W = \mathbb{C}v$

Have $\varphi_h(v) = \lambda_h v \in W$

$\Rightarrow$ W is invariant subspace

so $\dim V = 1$ ——

Cor: If G abelian, $\varphi: G \to GL_n(\mathbb{C})$

then $\exists$ $P \in GL_n(\mathbb{C})$ such that

$P^{-1}\varphi_g P$ is diagonal for all $g \in G$

Cor: If $A \in GL_n(\mathbb{C})$, $\underset{n}{\underset{\shortparallel}{o(A)}} < \infty$.

then A is diagonalizable.

Proof: $\mathbb{Z}_n \xrightarrow{\varphi} GL_n(\mathbb{C})$, $\varphi_{[k]} = A^k$

use complete reducibility ——

G - finite abelian group.

$$\{\text{irreducible } G \text{ reps}\}/\text{equivalence} \Longleftrightarrow \{G \to \mathbb{C}^\times\}$$

G - group, commutator subgroup of G

$$[G,G] := \langle \underbrace{ghg^{-1}h^{-1}} \mid g,h \in G \rangle$$

<u>Facts</u>: (1) $[G,G] \trianglelefteq G$ normal

(2) $G/[G,G]$ abelian

(3) Every $\varphi : G \to A$, A abelian is such that $[G,G] \subseteq \ker(\varphi)$ and so factors as

$$\begin{tikzcd} G \arrow[r, two heads, "\pi"] \arrow[rr, bend right, "\varphi"'] & G/[G,G] \arrow[r, "\bar{\varphi}"] & A = \mathbb{C}^\times \end{tikzcd}$$

$$\mathbb{C}^\times = GL_1(\mathbb{C})$$

Prop: The 1-dimensional representations of G are all of the form $\varphi \circ \pi$:

$$G \xrightarrow{\pi} \underline{G/[G,G]} \xrightarrow{\varphi} GL_1(\mathbb{C}).$$

Example: $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

$[Q_8, Q_8] = \{\pm 1\}$

$\Rightarrow \quad Q_8/[Q_8, Q_8] \approx \underline{\mathbb{Z}_2 \times \mathbb{Z}_2}$

$\Rightarrow \quad Q_8$ has 4 1-dimensional representations

Exer: $\varphi : G \to GL(V)$

given $\theta : G \to \mathbb{C}^\times$

$\Rightarrow \quad \theta\varphi : G \to GL(V)$ is a representation

$$(\theta\varphi)_g = \theta_g \varphi_g$$

$\theta\varphi$ irreducible $\Leftrightarrow$ φ irreducible.

e.g. $\rho : S_n \to GL_n(\mathbb{C})$, $\mathrm{sgn} : S_n \to \mathbb{C}^\times$ $\Rightarrow$ $\mathrm{sgn} \cdot \rho : S_n \to GL_n(\mathbb{C})$